**Math 358 – Foundations of Mathematics** **Fall 2008**
**10-30-08 Class Notes**

| **Sections 4.1, 4.3, 4.4** |
|---|
| 4.1: 16 (use #2) |
| 4.3: 27, 37, 44 |
| 4.4: 58, 59 |

Equivalence Relations

**Definition**: The ***cartesian product*** of two sets $A$ and $B$ is the set $A \times B = \{(a,b) : a \in A \text{ and } b \in B\}$.

**Definition**: Let $S$ be a set. A ***relation on*** $S$ is any set of ordered pairs of elements of $S$. In other words, a relation is a subset of $S \times S$. The ***domain*** of the relation is the set of first coordinates and the ***range*** is the set of second coordinates. If $x \in S$ and $R$ is a relation on $S$, we define $R[x] = \{y \in S : (x,y) \in R\}$. If the relation is clear, we'll often just write this as $[x]$.

**Example 1**: Let $S = \mathbb{R}$. Define $R_1 = \{(x,y) \in \mathbb{R} \times \mathbb{R} : x^2 + y^2 = 25\}$. Of course, when we are dealing with $\mathbb{R} \times \mathbb{R}$, we can visualize a relation as a set of points in the plane; in this case a circle of radius 5. The domain is $-5 \le x \le 5$ as is the range. $[4] = \{-3,3\}$, $[3] = \{-4,4\}$, $[-5] = \{0\}$, and $[2] = \{-\sqrt{21}, \sqrt{21}\}$.

**Example 2**: Let $S = \mathbb{N}$. Define $R_2 = \{(x,y) \in \mathbb{N} \times \mathbb{N} : y \le x \text{ and } \gcd(x,y) > 1\}$. This example does not lend itself to visualization like the previous one. However, we can still compute $[x]$ for many values of $x$. $[4] = \{2,4\}$, $[7] = \{7\}$, and $[30] = \{2,3,4,5,6,8,9,10,12,14,15,16,18,20,21,22,24,25,26,27,28,30\}$.

These examples show us that relations can have a few properties. For example, in Example 2, every element is related to itself. In other words, $(x,x) \in R_2$ for all $x \in \mathbb{N}$. But this is not true in Example 1. In Example 1, we can see that if $(x,y) \in R_1$, then $(y,x) \in R_1$. But this isn't true for $R_2$. These properties (and one more) have names.

**Definition**: Let $S$ be a nonempty set and let $R$ be a relation on $S$.

(a) If $(x,x) \in R$ for all $x \in S$, then $R$ is ***reflexive***.

(b) If $(x,y) \in R$ implies that $(y,x) \in R$, then $R$ is ***symmetric***.

(c) If $(x,y)\in R$ and $(y,z)\in R$ imply that $(x,z)\in R$, then $R$ is ***transitive***.

A relation that is reflexive, symmetric and transitive is called an ***equivalence relation*** on $S$.

Based on the previous discussion, we can see that the relations given in Examples 1 and 2 are not equivalence relations (since $R_1$ wasn't reflexive and $R_2$ wasn't symmetric). But just for kicks, let's check them for transitivity. If $(x,y)\in R_1$ and $(y,z)\in R_1$, does that imply that $(x,z)\in R_1$? In other words, if $x^2+y^2=25$ and $y^2+z^2=25$, then does $x^2+z^2$ necessarily equal 25? Nope. (For instance, $3^2+4^2=25$ and $4^2+3^2=25$, but $3^2+3^2\neq 25$.) How about $R_2$? If $(x,y)\in R_2$ and $(y,z)\in R_2$, does that imply that $(x,z)\in R_2$? If $x$ and $y$ have a common factor and $y$ and $z$ have a common factor, does that imply that $x$ and $z$ have a common factor? Nope again. (For instance $(8,6)\in R_2$ and $(6,3)\in R_2$, but $(8,3)\notin R_2$.)

We can restate the equivalence relation properties using the bracket notation as follows:

*A relation $R$ on a set $S$ is an equivalence relation if and only if:*

*(a) For each $x\in S$, $x\in[x]$.*

*(b) If $y\in[x]$, then $x\in[y]$.*

*(c) If $y\in[x]$ and $z\in[y]$, then $z\in[x]$.*

**Example 3**: Let $R=\{(x,y)\in\mathbb{R}\times\mathbb{R}: x=y\}$. Show that $R$ an equivalence relation on $\mathbb{R}$.

(a) Since $x=x$, $x\in[x]$ for all $x\in\mathbb{R}$.

(b) If $x=y$, then $y=x$. So $y\in[x]$ implies $x\in[y]$ implies $x\in[y]$.

(c) If $x=y$ and $y=z$, then $x=z$.

**Example 4**: Let $R=\{(x,y)\in\mathbb{R}\times\mathbb{R}: xy\geq 0\}$. Is $R$ an equivalence relation on $\mathbb{R}$?

(a) Since $x^2\geq 0$, $x\in[x]$ for all $x\in\mathbb{R}$.

(b) Since multiplication is commutative, if $xy\geq 0$, then $yx\geq 0$. So $y\in[x]$

implies $x \in [y]$.
(c) Nope. $0 \in [5]$ and $-3 \in [0]$, but $-3 \notin [5]$.

The set $[x]$ is called the ***equivalence class*** of $x$. We can think of an equivalence relation $R$ as a way to organize the elements of $S$ into disjoint classes. By the reflexive property, every element of $S$ is in some equivalence class. To see that the equivalence classes are disjoint, first notice that if $y \in [x]$, then $[y] \subseteq [x]$. Now, suppose that $y \in [x]$ and $y \in [z]$. (We want to show that $[x] = [z]$.) By symmetry of $R$, we have $z \in [y]$. Then by transitivity, $z \in [x]$. So $[z] \subseteq [x]$. But again, by symmetry, $x \in [z]$, and therefore $[x] \subseteq [z]$. So $[x] = [z]$. This notion of breaking up a set into disjoint subsets has a name also.

**Definition**: Let $S$ be a set. A pairwise disjoint collection $P$ of nonempty subsets of $S$ for which $\cup P = S$ is called a ***partition*** of $S$.

By the above discussion, $\{[x]\}$ is a partition of $S$.

**Example 5**: Let $S = \{1,2,3,\ldots 10\}$. Then $P = \{\{1,4,5\},\{2\},\{8,9,10\},\{3,6,7\}\}$ is a partition.

Notice that elements of a partition are subsets of the given set. And they are either disjoint or identical. So to show two partition sets are equal, you only have to show they have ONE element in common.

So we have seen that every equivalence relation on a set $S$ produces a partition of $S$. Let's see that every partition of $S$ produces an equivalence relation.

**Example 6**: Recall Example 5. Show that $R = \{(x, y) : x \text{ and } y \text{ are in the same partition set of } P\}$ is an equivalence relation on $S = \{1,2,3,\ldots 10\}$.

(a) Clearly $x$ is always in the same subset as itself.
(b) If $x$ and $y$ are in the same partition set (i.e. $(x, y) \in R$), then obviously, $y$ and $x$ are in the same partition set. So $(y, x) \in R$.
(c) If $x$ and $y$ are in the same set and $y$ and $z$ are in the same set, then $x$ and $z$ are in the same set.

One last issue to think about: Suppose we start with an equivalence relation $R$ on a set $S$. Related to this is a partition $P_R$. We just saw that related to any partition is an equivalence relation. If we find the equivalence relation generated by $P_R$, must it equal $R$? (We could ask the same question in reverse: is the partition associated to a equivalence relation that came from a partition the same as the original partition?)

Thankfully, the answer to both questions is yes.